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ON THE LEAST STRONGLY COMPACT CARDINAL[†]

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ABSTRACT

We prove that under the assumption of a supercompact cardinal κ which is a limit of supercompact cardinals, for any increasing Σ_2 function ϕ the set $\{\partial < \kappa : \partial$ is at least $\phi(\partial)$ supercompact, is strongly compact, yet is not fully supercompact$\}$ is unbounded in κ. We then use ideas of Magidor to show that under the hypotheses of a supercompact cardinal which is a limit of supercompact cardinals it is consistent for the least strongly compact cardinal κ_0 to be at least $\phi(\kappa_0)$ supercompact yet not to be fully supercompact, where ϕ is again an increasing Σ_2 function which also meets certain other technical restrictions.

The notions of strongly compact cardinal and supercompact cardinal have been studied quite intensely by set theorists over the last 10 years or so. Originally introduced as generalizations of the compactness theorem of first order logic to the infinitary language $L_{\kappa,\kappa}$ and of measurability (Keisler–Tarski [2] and Solovay–Reinhardt–Kanamori [9]), these cardinals have proven to be of immense interest. Indeed, many powerful theorems (see Kanamori–Magidor [1], for instance) have been proven assuming either their consistency or outright existence.

One reason that strongly compact cardinals are of interest is their strange behavior in the hierarchy of large cardinals. It was originally conjectured by Solovay that every strongly compact cardinal was supercompact. This conjecture is of course now known to be false. Menas in his thesis [6] showed that, assuming the existence of a measurable limit of strongly compact cardinals, there is a model in which there is exactly one stongly compact κ which is not even κ^+ supercompact, and that (*) assuming the existence of a cardinal κ which is a supercompact limit of supercompact cardinals, $\{\alpha < \kappa : \alpha$ is strongly compact yet

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is not 2^α supercompact} is unbounded in κ. Magidor then later substantially improved on Menas' results and showed [5]:

(1) Con (ZFC + There is a supercompact cardinal) $\Rightarrow$ Con(ZFC + The least supercompact cardinal is the least strongly compact cardinal).

(2) Con(ZFC + There is a strongly compact cardinal) $\Rightarrow$ Con(ZFC + The least strongly compact cardinal is the least measurable cardinal).

Thus, Magidor's results show that it is not possible to determine just exactly how "big" the least strongly compact cardinal is; it can either be the least supercompact cardinal (in which case, by a theorem of Solovay–Reinhardt [9] it is much bigger than the least measurable cardinal), or it can be the least measurable cardinal (in which case it is somewhat "small").

This paper studies further just what is consistent to assume about strongly compact cardinals in general, and the least strongly compact cardinal in particular. Specifically, we prove the following two theorems.

THEOREM 1. *Let ϕ be a formula in the language of set theory which defines an increasing Σ_2 function from the ordinals to the ordinals. Let κ be a supercompact limit of supercompact cardinals. Then $A = \{\partial < \kappa : \partial$ is strongly compact, at least $\phi(\partial)$ supercompact, yet is not fully supercompact$\}$ is unbounded in κ.*

THEOREM 2. *Assume that $V \models$ "ZFC + There is a supercompact limit of supercompact cardinals". Let ϕ be a formula in the language of set theory which defines an increasing Σ_2 function from the ordinals to the ordinals which, in addition, has the following properties:*

(1) ϕ is preserved above κ when forcing with a cardinal preserving partial ordering of size κ, i.e., for G V-generic on P, $\bar{\bar{P}} = \kappa$, if $\alpha > \kappa$, then $V \models$ "$\partial = \phi(\alpha)$" iff $V[G] \models$ "$\partial = \phi(\alpha)$".

(2) If $\alpha < \beta$, α is $\phi(\alpha)$ supercompact and β is $\phi(\beta)$ supercompact, then $\phi(\alpha) < \beta$.

Then it is consistent for the least strongly compact cardinal κ_0 to be at least $\phi(\kappa_0)$ supercompact yet not to be fully supercompact.

Restrictions (1) and (2) above on ϕ are made for technical reasons. Note, however, that many Σ_2 functions meet these restrictions. Some examples are $\partial \mapsto \partial^+$, $\partial \mapsto$ The least inaccessible $> \partial$, $\partial \mapsto$ The least measurable $> \partial$, etc.

The question of whether or not a strongly compact cardinal which is partially supercompact yet not fully supercompact could exist was first put to the author by Gerald Sacks. The above two theorems show that not only is this possible, but that under certain circumstances such cardinals are plentiful. Note also that

Theorem 2 is in some sense an "intermediate" result to the results of Menas and Magidor mentioned earlier; the least strongly compact cardinal can be partially supercompact yet not fully supercompact, in contrast to the least strongly compact not being supercompact at all (Menas) or to the least strongly compact being the least supercompact (Magidor).

Before beginning the proofs of Theorems 1 and 2, we briefly mention some beackground information. Basically, our notation and terminology are fairly standard. We work throughout in ZFC. Lower case Greek letters $\alpha, \beta, \gamma, \cdots$ denote ordinals, with κ, ∂ and λ generally being reserved for cardinals. $R(\alpha)$ denotes the collection of all sets of rank $< \alpha$, and V denotes the universe. For x a set, $\bar{\bar{x}}$ denotes the cardinality of x, and $2^{\bar{\bar{x}}}$ denotes the power set of x. For $\alpha < \beta$, $[\beta]^{<\alpha} = \bigcup_{\partial < \alpha} \{f : f$ is a strictly increasing function from ∂ to $\beta\}$.

When we talk about forcing, $\Vdash$ will mean, as usual, "weakly forces" and $\|$ will mean "decides". $p \leqq q$ means that q is *stronger* than p.

Occasionally we will be concerned with a formula ϕ which may define a non-absolute function. In this case, we write ϕ^V to mean ϕ as defined in V.

We presume that the reader is quite familiar with the notions of measurable cardinal, strongly compact cardinal, and supercompact cardinal. In particular, we will frequently use interchangeably the ultrafilter definition and embedding definition for λ supercompactness. For definitions and facts about these cardinals, we refer the reader to [1] and [9].

We now turn our attention to the proofs of Theorems 1 and 2.

PROOF OF THEOREM 1. Assume towards a contradiction that the set A is bounded in κ. Let $\beta < \kappa$ be an ordinal $> \sup A$, and let κ_0 be the least supercompact limit of supercompact cardinals $> \beta$.

Let $\gamma = \phi(\kappa_0)$. Let $\lambda \geqq \gamma$ be least such that for any inner model M of ZFC which is closed under λ sequences, $M \models$ "$\gamma = \phi(\kappa_0)$". Such a λ exists since ϕ is Σ_2.

Since $V \models$ "κ_0 is supercompact", let $j : V \to M$ be an elementary embedding such that κ_0 is the least ordinal moved by j, and $M^{2^{|\lambda|^{<\kappa_0}}} \subseteq M$. The closure properties of M ensure that $M \models$ "$\gamma = \phi(\kappa_0)$" and $M \models$ "κ_0 is at least $\phi(\kappa_0)$ supercompact". And, if $\theta < \kappa_0$ is a supercompact cardinal, by elementariness, $M \models$ "$j(\theta)$ is a supercompact cardinal", i.e., $M \models$ "θ is a supercompact cardinal".

Now $M \models$ "κ_0 is measurable" (since $M \models$ "κ_0 is $\phi(\kappa_0)$ supercompact"), and since the set of supercompacts below κ_0 in V is unbounded, the argument just given shows that the set of supercompacts below κ_0 in M is unbounded. Hence, by Menas' theorem that a measurable limit of strongly compact cardinals is strongly compact [6], $M \models$ "κ_0 is strongly compact".

Now $V \models$ "κ_0 is the least supercompact limit of supercompact cardinals $> \beta$". Hence, again by elementariness, $M \models$ "$j(\kappa_0)$ is the least supercompact limit of supercompact cardinals $> \beta$ $(= j(\beta))$", and $j(\kappa_0) > \kappa_0$. Hence, $M \models \Psi$, where Ψ is the formula stating "There exists a cardinal $\partial > \beta$ such that ∂ is less than the least supercompact limit of supercompact cardinals $> \beta$, ∂ is strongly compact, at least $\phi(\partial)$ supercompact, yet is not fully supercompact". Thus, by elementariness, $M \models \Psi$, contradicting the choice of β. This contradiction then proves Theorem 1. □ Theorem 1

We note that ϕ cannot in general by Σ_3. For example, if ϕ is the Σ_3 function which sends an ordinal α to the least supercompact cardinal $> \alpha$, then a theorem of Magidor [4] shows that the set A is empty. We also note that ϕ may imply additional hypotheses. For example, if ϕ says "Send α to the least strongly inaccessible $> \alpha$", then there is assumed to be a $\partial > \kappa$ which is strongly inaccessible.

It is also possible, under the correct hypotheses, for each $\partial \in A$ to be $\phi(\partial)$ supercompact yet not $(\phi(\partial))^+$ supercompact. For example, if we assume GCH, then the argument given in Theorem 1 shows that, for ϕ defined as the function $\partial \mapsto \partial^+$, it is possible for each $\partial \in A$ to be ∂^+ supercompact, strongly compact, yet not ∂^{++} supercompact. To do this, the embedding j of Theorem 1 is chosen so that κ_0 is the least ordinal moved, $j(\kappa_0)$ is minimal, and $M^{\kappa_0^+} \subseteq M$. A result of Menas [6] then shows that κ_0 will be κ_0^+ supercompact yet will not be κ_0^{++} supercompact.

We now turn over our attention to the proof of Theorem 2. The proof will use Magidor's notion of iterated Prikry forcing [5] for destroying the measurability of each element of a given set B of measurable cardinals, and we assume that the reader is familiar with this notion of forcing and its properties. We also assume that the reader is familiar with the notation and terminology of [5]; in particular, we will frequently use the distance function $| \quad |$ of [5].

Roughly speaking, the idea behind the proof of Theorem 2 is as follows: For any ϕ as in the hypotheses, we will let κ_0 be such that κ_0 is $\phi(\kappa_0)$ supercompact, not fully supercompact, yet fully strongly compact. We will then define an iterated Prikry ordering which will destroy all measurable cardinals $\partial < \kappa_0$ which are, either in V or in some generic extension of V, $\phi(\partial)$ supercompact, thus ensuring that κ_0 will not be $2^{[\phi(\kappa_0)]^{<\kappa_0}}$ supercompact. This set of cardinals, though, will be "thin" enough so that κ_0 is still $\phi(\kappa_0)$ supercompact in the generic extension. κ_0 will remain strongly compact for the same reasons as in [5], and κ_0

will be the least strongly compact since we will create an unbounded set of singular strong limit cardinals each of which violates GCH, so by Solovay's Theorem [8], there can be no strongly compacts below κ_0.

PROOF OF THEOREM 2. Let $V \models$ "ZFC + κ is a supercompact limit of supercompact cardinals", and assume that κ is the least such cardinal. Assume further that, in V, $2^{\partial} = \partial^{++}$ for ∂ inaccessible and $2^{\partial} = \partial^{+}$ otherwise; that this is possible is a theorem of Menas [7].

Let ϕ be as in the hypotheses of Theorem 2. Define two Σ_2 functions Ψ_1 and Ψ_2 as follows: $\Psi_1(\alpha)$ = the least $\beta \geqq \alpha$ such that $\langle R(\beta), \in \rangle \models$ "$\gamma = \phi(\alpha)$", i.e., β is the least ordinal so that $\langle R(\beta), \in \rangle$ captures the definition of $\phi(\alpha)$, and let $\Psi_2(\alpha) = \overline{\overline{R(\Psi_1(\alpha))}}$.

Let $k : V \to N$ be an elementary embedding so that κ is the least ordinal moved and so that $N^{[\psi_2(\kappa)]^{<\kappa}} \subseteq N$. By the definition of N and ψ_2, we will have $N \models$ "κ is $\psi_2(\kappa)$ supercompact"; by the proof of Theorem 1, as κ has been chosen to be the least supercompact limit of supercompacts, $N \models$ "κ is not supercompact". As k is elementary and fixes each $\partial < \kappa$, the fact that in V there are unboundedly many $\partial < \kappa$ which are supercompact implies that in N there are unboundedly many $\partial < \kappa$ which are supercompact. κ is thus a measurable limit of supercompacts in N, so by Menas' theorem [6] that a measurable limit of strongly compacts is strongly compact, $N \models$ "κ is strongly compact". Also, each $\partial < \kappa$ which is supercompact (in either V or N) is obviously $\phi(\partial)$ supercompact, by the definition of ϕ. Thus, in N there is a cardinal, κ, which is $\psi_2(\kappa)$ supercompact, strongly compact, not fully supercompact, and is a limit of strongly compact cardinals ∂ which are $\phi(\partial)$ supercompact. Hence as V and N are elementarily equivalent let, in V, κ_0 be such that κ_0 is $\psi_2(\kappa_0)$ supercompact, strongly compact, not fully supercompact, with the set $B = \{\partial < \kappa : \partial$ is $\phi(\partial)$ supercompact, ∂ is strongly compact$\}$ unbounded in κ_0, and fix $j : V \to M$ an elementary embedding so that κ_0 is the least ordinal moved and so that $M^{\psi_2(\kappa_0)} \subseteq M$. Note that by the definition of ψ_2, $M \models$ "κ_0 is $\phi(\kappa_0)$ supercompact".

We define inductively on $\alpha < \kappa_0$ our partial ordering P. First, for each measurable $\beta < \kappa_0$, choose U_β a normal measure on β which gives measure 0 to the set of measurables $< \beta$. Next, we define a partial ordering P_α and set B_α. P_0 is the trivial partial ordering, and $B_0 = \phi$. For $\alpha > 0$, we let $C_\alpha = \bigcup_{\beta < \alpha} B_\alpha$, and let $c(\alpha)$ = the least cardinal $\partial \geqq \bigcup_{\beta \in C_\alpha} \phi(\beta)$ such that for some p in the iterated Prikry ordering Q_α which destroys all the measurables in C_α and is defined using, for $\beta \in C_\alpha$, the canonical $\tilde{U}_\beta$ which extends U_β (see [5]), $p \Vdash$ "∂ is $\phi(\partial)$ supercompact". B_α is then defined as $C_\alpha \cup \{\partial\}$, and P_α is defined as the iterated

Prikry ordering which destroys all the measurables in B_α using, for $\beta \in B_\alpha$, the canonical $\tilde{U}_\beta$ which extends U_β. Since forcing with an iterated Prikry ordering creates no new measurable cardinals [5], this definition makes sense. Finally, define P as the iterated Prikry ordering which destroys all the measurables in C_{κ_0} using Q_{κ_0}.

Let G be V-generic on P. We use ideas of Magidor [5] to show that, when forcing with P, κ_0 is $\phi(\kappa_0)$ supercompact, not fully supercompact, and is the least strongly compact cardinal. Note first that in M, $j(P)$ is a partial ordering defined through stage $j(\kappa_0)$ in the same manner that P was and P is an "initial segment" of $j(P)$, i.e., $j(P) = P * Q$, where Q is a term in M for the definition of $j(P)$ from stages κ_0 to $j(\kappa_0)$, and $*$ is as in [10].

Consider now what happens in M at stage κ_0 of the inductive definition of $j(P)$. There are κ_0 many distinct cardinals in C_{κ_0}; if there were fewer than κ_0 cardinals in C_{κ_0}, then let $\beta < \kappa_0$ be their sup (the fact that ϕ is Σ_2 ensures that $\beta < \kappa_0$). Then $P = Q_\alpha$ for some $\alpha < \kappa_0$, so by the techniques of Lévy–Solovay [3] and property (1) of ϕ, any element of $B > \beta$ retains its $\phi(\beta)$ supercompactness. This means that $P \neq Q_\alpha$. Thus, by property (2) of ϕ, $\bigcup_{\beta \in C_{\kappa_0}} \phi(\beta) = \kappa_0$, so $c^M(\kappa_0) \geqq \kappa_0$.

Case 1. $c^M(\kappa_0) > \kappa_0$. In this case, we know that $c^M(\kappa_0) > \phi^M(\kappa_0)$. This is seen as follows: If $c^M(\kappa_0) \leqq \phi^M(\kappa_0)$, then there would be in $M[H]$, for some H M-generic on P, a normal ultrafilter on $P^{M[H]}_{c^M(\kappa_0)}(\phi^{M[H]}(c^M(\kappa_0)))$. $c^M(\kappa_0)$ is thus a measurable cardinal in $M[H]$. However, as forcing with P creates no new measurable cardinals [5], $c^M(\kappa_0)$ is actually measurable in M.

Now $\bar{\bar{P}}^M \leqq (2^\kappa)^M < c^M(\kappa_0)$ since $c^M(\kappa_0)$ is thus strongly inaccessible in M; hence, again by the Lévy–Solovay results on mild Cohen extensions [3], there is actually a normal ultrafilter $U \in M$ on $P^M_{c^M(\kappa_0)}(\phi^{M[H]}(c^M(\kappa_0)))$. But property (1) of ϕ implies that $\phi^{M[H]}(c^M(\kappa_0)) = \phi^M(c^M(\kappa_0))$, so U is actually a normal ultrafilter on $P^M_{c^M(\kappa_0)}(\phi^M(\kappa_0))$. But we know that in M, κ_0 is $\phi^M(\kappa_0)$ supercompact, $c^M(\kappa_0)$ is $\phi^M(c^M(\kappa_0))$ supercompact, $c^M(\kappa_0) \leqq \phi^M(\kappa_0)$, and $\kappa_0 < c^M(\kappa_0)$. This, however, immediatly contradicts property (2) of ϕ.

We use Magidor's methods [5] to show that, in $V[G]$, $P^{V[G]}_{\kappa_0}(\phi^{V[G]}(\kappa_0))$ carries a normal ultrafilter. As in [5], define a normal ultrafilter $\tilde{U}$ over $P^{V[G]}_{\kappa_0}(\phi^{V[G]}(\kappa_0))$ as follows:

$$p \Vdash \text{“}\tau \in \tilde{U}\text{”} \text{ iff } p \Vdash \text{“}\tau \subseteq P^{V[G]}_{\kappa_0}(\phi^{V[G]}(\kappa_0))\text{”}$$

and there is $q \geqq j(p)$ in $j(P)$ such that $|j(p) - q| = 0$, $q \restriction c^M(\kappa_0) = j(p) \restriction c^M(\kappa_0) = p$, and $q \Vdash$ "$\langle j(\alpha) : \alpha < \phi^{V[G]}(\kappa_0)\rangle \in j(\tau)$". As $\phi^M(\kappa_0) = \phi^V(\kappa_0) = \phi^{V[G]}(\kappa_0)$ by choice of j and M and by property (1) of ϕ, this will define a normal ultrafilter on $P_{\kappa_0}^{V[G]}(\phi^{V[G]}(\kappa_0))$.

Now as in [5], we can show that $\tilde{U}$ always denotes a κ_0-additive fine measure on $P_{\kappa_0}^{V[G]}(\phi^{V[G]}(\kappa_0))$. To show that $\tilde{U}$ is normal, we let $p \Vdash$ "τ is a function from $P_{\kappa_0}^{V[G]}(\phi^{V[G]}(\kappa_0))$ into $\phi^{V[G]}(\kappa_0)$ such that $\tau(s) \in s$ for every $s \in P_{\kappa_0}^{V[G]}(\phi^{V[G]}(\kappa_0))$". We then show that p can be extended to a condition q such that for some $\sigma < \phi^{V[G]}(\kappa_0)$, $q \Vdash$ "$\{s : \tau(s) = \sigma\} \in \tilde{U}$".

To prove this last statement, let $\langle \chi_\sigma : \sigma < \phi^{V[G]}(\kappa_0)\rangle$ be the sequence of statements in the forcing language appropriate for $j(P)$ defined as follows: $\chi_\sigma \equiv j(\tau)(\langle j(\alpha) : \alpha < \phi^{V[G]}(\kappa_0)\rangle) = j(\sigma)$. Since $\phi^{V[G]}(\kappa_0) = \phi^V(\kappa_0) \leqq \Psi_2(\kappa_0)$, and $M^{\Psi_2(\kappa_0)} \subseteq M$, the sequence $\langle \chi_\sigma : \sigma < \phi^{V[G]}(\kappa_0)\rangle \in M$. Also, $c^M(\kappa_0)$ is not a limit point of $j(P)$, since $\bigcup_{\beta \in C_{\kappa_0}} \phi(\beta) = \kappa_0$, $c^M(\kappa_0) > \kappa_0$, $c^M(\kappa_0) > \phi^M(\kappa_0) = \phi^{V[G]}(\kappa_0)$. Hence, in M, we may apply lemma 2.4 of [5] for $j(p)$, $c^M(\kappa_0)$, and the sequence $\langle \chi_\sigma : \sigma < \phi^{V[G]}(\kappa_0)\rangle$ and obtain $q \in j(P)$, $q \geqq j(p)$ such that:

(1) $q \restriction c^M(\kappa_0) = j(p) \restriction c^M(\kappa_0) = p$.

(2) $|j(p) - q| = 0$.

(3) If $q \leqq q'$, $q' \| \chi_\sigma$, then $q' \restriction c^M(\kappa_0) \cup (q - p) \| \chi_\sigma$.

Now $j(p) \Vdash$ "$j(\tau)$ is a function from $P_{j(\kappa_0)}^{M[G']}(j(\phi^{V[G]}(\kappa_0)))$ into $j(\phi^{V[G]}(\kappa_0))$ such that $j(\tau)(s) \in s$ for every $s \in P_{j(\kappa_0)}^{M[G']}(j(\phi^{V[G]}(\kappa_0)))$". Hence, as

$$\langle j(\alpha) : \alpha < \phi^{V[G]}(\kappa_0)\rangle \in P_{j(\kappa_0)}^{M[G']}(j(\phi^{V[G]}(\kappa_0))) \quad (j(\kappa_0) > \phi^{V[G]}(\kappa_0)),$$

we can find a $q' \geqq q$ such that $q' \Vdash$ "$j(\tau)(\langle j(\alpha) : \alpha < \phi^{V[G]}(\kappa_0)\rangle) = j(\sigma)$" for some $\sigma < \phi^{V[G]}(\kappa_0)$. But by definition of q, $r = q' \restriction c^M(\kappa_0) \cup (q - p) \Vdash \chi_\sigma$, i.e.,

$$r \Vdash \text{“}\langle j(\alpha) : \alpha < \phi^{V[G]}(\kappa_0)\rangle \in \{s \in P_{j(\kappa_0)}^{M[G']}(j(\phi^{V[G]}(\kappa_0))) : j(\tau)(s) = j(\sigma)\}\text{”},$$

i.e.,

$$r \Vdash \text{“}\langle j(\alpha) : \alpha < \phi^{V[G]}(\kappa_0)\rangle \in j(\{s \in P_{\kappa_0}^{V[G]}(\phi^{V[G]}(\kappa_0)) : \tau(s) = \sigma\})\text{”}.$$

By a similar argument to theorem 2.5 of [5], using the above, $p \leqq q' \restriction c^M(\kappa_0)$, $r \restriction c^M(\kappa_0) = q' \restriction c^M(\kappa_0)$, and

$$|j(q' \restriction c^M(\kappa_0)) - r \restriction c^M(\kappa_0)| = 0.$$

Hence, by definition of $\tilde{U}$, $p \leqq q' \restriction c^M(\kappa_0) \Vdash$ "$\{s \in P_{\kappa_0}^{V[G]}(\phi^{V[G]}(\kappa_0)) : \tau(s) = \sigma\} \in \tilde{U}$". This shows that $\tilde{U}$ is a normal ultrafilter in $V[G]$ over $P_{\kappa_0}^{V[G]}(\phi^{V[G]}(\kappa_0))$.

Since $V \models$ "κ_0 is strongly compact", Magidor's argument of [5] shows that $V[G] \models$ "κ_0 is strongly compact". And, as remarked earlier, there are κ_0 elements ∂ in C_{κ_0}. Each of these ∂ is measurable in V, so since $V \models$ "$2^\partial = \partial^{++}$"

for ∂ strongly inaccessible, in $V[G]$, there is below κ_0 an unbounded set of singular strong limit cardinals each of which violates GCH. Hence, by Solovay's theorem [8] that GCH must hold for any singular strong limit cardinal above a strongly compact cardinal, there can be in $V[G]$ no strongly compact cardinals $< \kappa_0$. Therefore, $V[G] \models$ "κ_0 is the least strongly compact cardinal".

Finally, we show that in $V[G]$, κ_0 is not fully supercompact. If $V[G] \models$ "κ_0 is supercompact", let k be an elementary embedding of $V[G]$ into a sufficiently closed inner model N so that κ_0 is the least ordinal moved and so that $N \models$ "κ_0 is $\phi(\kappa_0)$ supercompact". By a standard reflection argument, we then have that $\{\partial < \kappa_0 : \partial$ is $\phi(\partial)$ supercompact$\}$ is unbounded in κ_0. Let ∂ be the least cardinal below κ_0 in $V[G]$ which is $\phi(\partial)$ supercompact, and let P_∂ be the portion of P which destroys all the measurables in $C_{\kappa_0} \cap \partial$. P can be viewed as $P_\partial * P^\partial$ where P^∂ destroys all the measurables in $C_{\kappa_0} \geqq \partial$, and G can be viewed as $G_\partial * G^\partial$, where G_∂ is V-generic on P and G^∂ is $V[G_\partial]$ generic on P^∂. Thus, $V[G_\partial][G^\partial] \models$ "∂ is $\phi(\partial)$ supercompact".

Now $V[G_\partial]$ must be a model of "∂ is $\phi(\partial)$ supercompact"; to see this, let λ be the least measurable in the field of P^∂. Lemma 2.1 of [5] tells us that, by the same arguments as in ordinary Prikry forcing, forcing with P^∂ adds no new bounded subsets to λ. We must have $\phi^{V[G]}(\partial) < \lambda$, for if $\phi^{V[G]}(\partial) \geqq \lambda$, then as $V[G] \models$ "$\mathrm{cof}(\lambda) = \omega$", ∂ cannot possibly be $\phi^{V[G]}(\partial)$ supercompact in $V[G]$. Hence, putting the last two sentences together, we get that $\phi^{V[G]}(\partial) = \phi^{V[G_\partial]}(\partial)$ and that if $V[G_\partial][G^\partial] \models$ "∂ is $\phi(\partial)$ supercompact", then $V[G_\partial] \models$ "∂ is $\phi(\partial)$ supercompact". But by the inductive definition of P, this immediately implies that $\lambda = \partial$, so in $V[G]$, $\mathrm{cof}(\lambda) = \omega$. Hence, there is no $\partial < \kappa_0$ which is $\phi(\partial)$ supercompact, so in $V[G]$, κ_0 is not supercompact. Thus, when $c^M(\kappa_0) > \kappa_0$, we have shown that in $V[G]$, κ_0 is $\phi(\kappa_0)$ supercompact, not fully supercompact, and is the least strongly compact cardinal. This completes the proof for Case 1.

Case 2. $c^M(\kappa_0) = \kappa_0$. In this case, we let H be M-generic on P so that $M[H] \models$ "κ_0 is $\phi(\kappa_0)$ supercompact". κ_0 is measurable in M since forcing with P creates no new measurable cardinals. j is elementary and fixes every $\partial < \kappa_0$; hence, as the set of strongly compacts below κ_0 in V is unbounded, the set of strongly compacts below κ_0 in M is unbounded. But by Menas' theorem [6] that a measurable limit of strongly compacts is strongly compact, $M \models$ "κ_0 is strongly compact". Thus, again as in [5], $M[H] \models$ "κ_0 is strongly compact". The argument given in Case 1 applies here also to show that, in $M[H]$, κ_0 is not supercompact and is the least strongly compact cardinal. Hence, when $c^M(\kappa_0) = \kappa_0$, $M[H] \models$ "κ_0 is $\phi(\kappa_0)$ supercompact, not supercompact, and is the least strongly compact cardinal". This completes the proof for Case 2, and also completes the proof of Theorem 2. □ Theorem 2

We remark that as with Theorem 1, it is possible to get sharp bounds on the non-supercompactness of κ_0 for certain ϕ. For example, if ϕ is the function $\alpha \mapsto \alpha^+$, then it is possible for κ_0 to be κ_0^+ supercompact, not κ_0^{++} supercompact, and be the least strongly compact cardinal. This will follow since forcing with P will preserve the fact that $2^{\kappa_0} = \kappa_0^{++}$.

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